The

Foundations of Arithmetic

A logico-mathematical enquiry into the concept of number

GOTTLOB FREGE

English Translation by
J. L. AUSTIN, M.A.

Second Revised Edition

NORTHWESTERN UNIVERSITY PRESS
EVANSTON · ILLINOIS · 1980

INTRODUCTION

When we ask someone what the number one is, or what the symbol 1 means,* we get as a rule the answer "Why, a thing". And if we go on to point out that the proposition

"the number one is a thing"

is not a definition, because it has the definite article on one side and the indefinite on the other, or that it only assigns the number one to the class of things, without stating which thing it is, then we shall very likely be invited to select something for ourselves—anything we please—to call one. Yet if everyone had the right to understand by this name whatever he pleased, then the same proposition about one would mean different things for different people,—such propositions would have no common content. Some, perhaps, will decline to answer the question, pointing out that it is impossible to state, either, what is meant by the letter a, as it is used in arithmetic; and that if we were to say "a means a number," this would be open to the same objection as the definition "one is a thing." Now in the case of a it is quite right to decline to answer: a does not mean some one definite number which can be specified, but serves to express the generality of general propositions. If, in $a + a - a = a$, we put for a

* [I have tried throughout to translate *Bedeutung* and its cognates by "meaning" and *Sinn* and its cognates by "sense", in view of the importance Frege later attached to the distinction. But it is quite evident that he attached no special significance to the words at this period.]

some number, any we please but the same throughout, we get always a true identity.* This is the sense in which the letter *a* is used. With one, however, the position is essentially different. Can we, in the identity $1 + 1 = 2$, put for 1 in both places some one and the same object, say the Moon? On the contrary, it looks as though, whatever we put for the first 1, we must put something different for the second. Why is it that we have to do here precisely what would have been wrong in the other case? Again, arithmetic cannot get along with *a* alone, but has to use further letters besides (*b*, *c* and so on), in order to express in general form relations between different numbers. It would therefore be natural to suppose that the symbol 1 too, if it served in some similar way to confer generality on propositions, could not be enough by itself. Yet surely the number one looks like a definite particular object, with properties that can be specified, for example that of remaining unchanged when multiplied by itself? In this sense, *a* has no properties that can be specified, since whatever can be asserted of *a* is a common property of all numbers, whereas $1^1 = 1$ asserts nothing of the Moon, nothing of the Sun, nothing of the Sahara, nothing of the Peak of Teneriffe; for what could be the sense of any such assertion?

Questions like these catch even mathematicians for that matter, or most of them, unprepared with any satisfactory answer. Yet is it not a scandal that our science should be so unclear about the first and foremost among its objects, and one which is apparently so simple? Small hope, then, that we shall be able to say what number is. If a concept fundamental to a mighty science gives rise to difficulties, then it is surely an imperative task to investigate it more closely until those difficulties are overcome; especially as we shall hardly succeed in finally clearing up negative numbers, or fractional or complex numbers, so long as our insight into the foundation of the whole structure of arithmetic is still defective.

* [*Gleichung*. This also means, and would often be more naturally translated, "equation". But I have generally retained "identity", because this is sometimes essential and because Frege does understand equations as identities. For similar reasons I have translated *gleich* "identical", though it can mean "equal" or even merely "similar". Cp. §§ 34, 65.]

Admittedly, many people will think this not worth the trouble. Naturally, they suppose, this concept is adequately dealt with in the elementary textbooks, where the subject is settled once and for all. Who can believe that he has anything still to learn on so simple a matter? So free from all difficulty is the concept of positive whole number held to be, that an account of it fit for children can be both scientific and exhaustive; and that every schoolboy, without any further reflexion or acquaintance with what others have thought, knows all there is to know about it. The first prerequisite for learning anything is thus utterly lacking—I mean, the knowledge that we do not know. The result is that we still rest content with the crudest of views, even though since HERBART's[1] day a better doctrine has been available. It is sad and discouraging to observe how discoveries once made are always threatening to be lost again in this way, and how much work promises to have been done in vain, because we fancy ourselves so well off that we need not bother to assimilate its results. My work too, as I am well aware, is exposed to this risk. A typical crudity confronts me, when I find calculation described as "aggregative mechanical thought".[2] I doubt whether there exists any thought whatsoever answering to this description. An aggregative imagination, even, might sooner be let pass; but that has no relevance to calculation. Thought is in essentials the same everywhere: it is not true that there are different kinds of laws of thought to suit the different kinds of objects thought about. Such differences as there are consist only in this, that the thought is more pure or less pure, less dependent or more upon psychological influences and on external aids such as words or numerals, and further to some

[1] Collected Works, ed. Hartenstein, Vol. X, part i, *Umriss pädagogischer Vorlesungen*, § 252, n. 2: "Two does not mean two things, but doubling" etc.

[2] K. Fischer, *System der Logik und Metaphysik oder Wissenschaftslehre*, 2nd edn., §94.

extent too in the finer or coarser structure of the concepts involved; but it is precisely in this respect that mathematics aspires to surpass all other sciences, even philosophy.

The present work will make it clear that even an inference like that from n to $n + 1$, which on the face of it is peculiar to mathematics, is based on the general laws of logic, and that there is no need of special laws for aggregative thought. It is possible, of course, to operate with figures mechanically, just as it is possible to speak like a parrot: but that hardly deserves the name of thought. It only becomes possible at all after the mathematical notation has, as a result of genuine thought, been so developed that it does the thinking for us, so to speak. This does not prove that numbers are formed in some peculiarly mechanical way, as sand, say, is formed out of quartz granules. In their own interests mathematicians should, I consider, combat any view of this kind, since it is calculated to lead to the disparagement of a principal object of their study, and of their science itself along with it. Yet even in the works of mathematicians are to be found expressions of exactly the same sort. The truth is quite the other way: the concept of number, as we shall be forced to recognize, has a finer structure than most of the concepts of the other sciences, even although it is still one of the simplest in arithmetic.

In order, then, to dispel this illusion that the positive whole numbers really present no difficulties at all, but that universal concord reigns about them, I have adopted the plan of criticizing some of the views put forward by mathematicians and philosophers on the questions involved. It will be seen how small is the extent of their agreement—so small, that we find one dictum precisely contradicting another. For example, some hold that "units are identical with one another," others that they are different, and each side supports its assertion with arguments that cannot be rejected out of hand. My object in this is

to awaken a desire for a stricter enquiry. At the same time this preliminary examination of the views others have put forward should clear the ground for my own account, by convincing my readers in advance that these other paths do not lead to the goal, and that my opinion is not just one among many all equally tenable; and in this way I hope to settle the question finally, at least in essentials.

I realize that, as a result, I have been led to pursue arguments more philosophical than many mathematicians may approve; but any thorough investigation of the concept of number is bound always to turn out rather philosophical. It is a task which is common to mathematics and philosophy.

It may well be that the co-operation between these two sciences, in spite of many démarches from both sides, is not so flourishing as could be wished and would, for that matter, be possible. And if so, this is due in my opinion to the predominance in philosophy of psychological methods of argument, which have penetrated even into the field of logic. With this tendency mathematics is completely out of sympathy, and this easily accounts for the aversion to philosophical arguments felt by many mathematicians. When STRICKER,[1] for instance, calls our ideas* of numbers motor phenomena and makes them dependent on muscular sensations, no mathematician can recognize his numbers in such stuff or knows what on earth to make such a proposition. An arithmetic founded on muscular sensations would certainly turn out sensational enough, but also every bit as vague as its foundation. No, sensations are absolutely no concern of arithmetic. No more are mental pictures, formed from the amalgamated traces of earlier sense-impressions. All these phases of consciousness are characteristically fluctuating and indefinite, in strong contrast to the definiteness and fixity of the concepts and objects of

[1] *Studien über Association der Vorstellungen*, Vienna 1883.

* [*Vorstellungen.* I have translated this word consistently by "idea", and cognate words by "imagine", "imagination", etc. For Frege it is a psychological term, cp. p. x^{e} below.]

mathematics. It may, of course, serve some purpose to investigate the ideas and changes of ideas which occur during the course of mathematical thinking; but psychology should not imagine that it can contribute anything whatever to the foundation of arithmetic. To the mathematician as such these mental pictures, with their origins and their transformations, are immaterial. STRICKER himself states that the only idea he associates with the word "hundred" is the symbol 100. Others may have the idea of the letter C or something else; does it not follow, therefore, that these mental pictures are, so far as concerns us and the essentials of our problem, completely immaterial and incidental—as incidental as chalk and blackboard, and indeed that they do not deserve to be called ideas of the number a hundred at all? Never, then, let us suppose that the essence of the matter lies in such ideas. Never let us take a description of the origin of an idea for a definition, or an account of the mental and physical conditions on which we become conscious of a proposition for a proof of it. A proposition may be thought, and again it may be true; let us never confuse these two things. We must remind ourselves, it seems, that a proposition no more ceases to be true when I cease to think of it than the sun ceases to exist when I shut my eyes. Otherwise, in proving Pythagoras' theorem we should be reduced to allowing for the phosphorous content of the human brain; and astronomers would hesitate to draw any conclusions about the distant past, for fear of being charged with anachronism,—with reckoning twice two as four regardless of the fact that our idea of number is a product of evolution and has a history behind it. It might be doubted whether by that time it had progressed so far. How could they profess to know that the proposition $2 \times 2 = 4$ already held good in that remote epoch? Might not the creatures then extant have held the proposition $2 \times 2 = 5$, from which the proposition $2 \times 2 = 4$ was only evolved later through a process of natural selection

in the struggle for existence? Why, it might even be that $2 \times 2 = 4$ itself is destined in the same way to develop into $2 \times 2 = 3$! *Est modus in rebus, sunt certi denique fines*! The historical approach, with its aim of detecting how things begin and of arriving from these origins at a knowledge of their nature, is certainly perfectly legitimate; but it has also its limitations. If everything were in continual flux, and nothing maintained itself fixed for all time, there would no longer be any possibility of getting to know anything about the world and everything would be plunged in confusion. We suppose, it would seem, that concepts sprout in the individual mind like leaves on a tree, and we think to discover their nature by studying their birth: we seek to define them psychologically, in terms of the nature of the human mind. But this account makes everything subjective, and if we follow it through to the end, does away with truth. What is known as the history of concepts is really a history either of our knowledge of concepts or of the meanings of words. Often it is only after immense intellectual effort, which may have continued over centuries, that humanity at last succeeds in achieving knowledge of a concept in its pure form, in stripping off the irrelevant accretions which veil it from the eyes of the mind. What, then, are we to say of those who, instead of advancing this work where it is not yet completed, despise it, and betake themselves to the nursery, or bury themselves in the remotest conceivable periods of human evolution, there to discover, like JOHN STUART MILL, some gingerbread or pebble arithmetic! It remains only to ascribe to the flavour of the bread some special meaning for the concept of number. A procedure like this is surely the very reverse of rational, and as unmathematical, at any rate, as it could well be. No wonder the mathematicians turn their backs on it. Do the concepts, as we approach their supposed sources, reveal themselves in peculiar purity? Not at all;

we see everything as through a fog, blurred and undifferentiated. It is as though everyone who wished to know about America were to try to put himself back in the position of Columbus, at the time when he caught the first dubious glimpse of his supposed India. Of course, a comparison like this proves nothing; but it should, I hope, make my point clear. It may well be that in many cases the history of earlier discoveries is a useful study, as a preparation for further researches; but it should not set up to usurp their place.

So far as mathematicians are concerned, an attack on such views would indeed scarcely have been necessary; but my treatment was designed to bring each dispute to an issue for the philosophers as well, as far as possible, so that I found myself forced to enter a little into psychology, if only to repel its invasion of mathematics.

Besides, even mathematical textbooks make use of psychological expressions. When the author feels himself obliged to give a definition, yet cannot, then he tends to give at least a description of the way in which we arrive at the object or concept concerned. These cases can easily be recognized by the fact that such explanations are never referred to again in the course of the subsequent exposition. For teaching purposes, introductory devices are certainly quite legitimate; only they should always be clearly distinguished from definitions. A delightful example of the way in which even mathematicians can confuse the grounds of proof with the mental or physical conditions to be satisfied if the proof is to be given is to be found in E. SCHRÖDER.[1] Under the heading "Special Axiom" he produces the following: "The principle I have in mind might well be called the Axiom of Symbolic Stability. It guarantees us that throughout all our arguments and deductions the symbols

[1] *Lehrbuch der Arithmetik und Algebra*, [Leipzig 1873].

remain constant in our memory—or preferably on paper," and so on.

No less essential for mathematics than the refusal of all assistance from the direction of psychology, is the recognition of its close connexion with logic. I go so far as to agree with those who hold that it is impossible to effect any sharp separation of the two. This much everyone would allow, that any enquiry into the cogency of a proof or the justification of a definition must be a matter of logic. But such enquiries simply cannot be eliminated from mathematics, for it is only through answering them that we can attain to the necessary certainty.

In this direction too I go, certainly, further than is usual. Most mathematicians rest content, in enquiries of this kind, when they have satisfied their immediate needs. If a definition shows itself tractable when used in proofs, if no contradictions are anywhere encountered, and if connexions are revealed between matters apparently remote from one another, this leading to an advance in order and regularity, it is usual to regard the definition as sufficiently established, and few questions are asked as to its logical justification. This procedure has at least the advantage that it makes it difficult to miss the mark altogether. Even I agree that definitions must show their worth by their fruitfulness: it must be possible to use them for constructing proofs. Yet it must still be borne in mind that the rigour of the proof remains an illusion, even though no link be missing in the chain of our deductions, so long as the definitions are justified only as an afterthought, by our failing to come across any contradiction. By these methods we shall, at bottom, never have achieved more than an empirical certainty, and we must really face the possibility that we may still in the end encounter a contradiction which brings the whole edifice down in ruins. For this reason I have felt bound to go back rather further into the general logical foundations of our science than perhaps most mathematicians will consider necessary.

In the enquiry that follows, I have kept to three fundamental principles:

always to separate sharply the psychological from the logical, the subjective from the objective;

never to ask for the meaning of a word in isolation, but only in the context of a proposition;

never to lose sight of the distinction between concept and object.

In compliance with the first principle, I have used the word "idea" always in the psychological sense, and have distinguished ideas from concepts and from objects. If the second principle is not observed, one is almost forced to take as the meanings of words mental pictures or acts of the individual mind, and so to offend against the first principle as well. As to the third point, it is a mere illusion to suppose that a concept can be made an object without altering it. From this it follows that a widely-held formalist theory of fractional, negative, etc., numbers is untenable. How I propose to improve upon it can be no more than indicated in the present work. With numbers of all these types, as with the positive whole numbers, it is a matter of fixing the sense of an identity.

My results will, I think, at least in essentials, win the adherence of those mathematicians who take the trouble to attend to my arguments. They seem to me to be in the air, and it may be that every one of them singly, or at least something very like it, has been already put forward; though perhaps, presented as they are here in connexion with each other, they may still be novel. I have often been astonished at the way in which writers who on one point approach my view so closely, on others depart from it so violently.

Their reception by philosophers will be varied, depending on each philosopher's own position; but presumably those

empiricists who recognize induction as the sole original process of inference (and even that as a process not actually of inference but of habituation) will like them least. Some one or another, perhaps, will take this opportunity to examine afresh the principles of his theory of knowledge. To those who feel inclined to criticize my definitions as unnatural, I would suggest that the point here is not whether they are natural, but whether they go to the root of the matter and are logically beyond criticism.

I permit myself the hope that even the philosophers, if they examine what I have written without prejudice, will find in it something of use to them.